Acta Cryst. (1973). A29, 673

The Hyper-Stereographic Projection of the Four-Dimensional Hyper-Sphere

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(*Received 5 March* 1973; *accepted* 1 *April* 1973)

The hyper-stereographic projection of the hyper-sphere is defined, and its properties are discussed. It is used to represent in a three-dimensional figure the effect of reflexion in a hyper-plane and rotation about a plane.

Introduction

The convenience of the stereographic projection for the representation in two dimensions of points on the surface of the sphere has been known at least since the time of Ptolemy (Commandinus, 1558), and its use for the equivalent problem of the manipulation of directions in space has been well known since its introduction into crystallography by Neumann (1823). Its usefulness depends on the much greater facility with which constructions may be visualized in two dimensions rather than in three. The still greater disparity in the ease of visualizing constructions in three dimensions rather than in four implies that a corresponding hyper-stereographic projection of the four-dimensional hyper-sphere could be of substantial assistance in considering the problems involved in rotations and symmetry operations applied to directions in four dimensions, and the current interest in four-dimensional symmetry suggests that such assistance may have some applications.

Definition of the hyper-stereographic projection

Let us first consider an analytic definition of the stereographic projection. If P is a point (x,y,z) on the surface of the sphere

$$x^2+y^2+z^2=1$$

and P' $(x,y,0)$ is the orthogonal projection of P on the plane $z=0$, then the stereographic projection of P with respect to the south pole $(0,0,-1)$ is a point on the plane $z=0$ with polar coordinates $(\tan\theta/2,\varphi)$ where θ is the angle $P\hat{O}Z$ and φ is the angle $P'\hat{O}Y$.

The hyper-stereographic projection may then be defined analogously. P is a point (w,x,y,z) on the boundary of the hyper-sphere

$$w^2+x^2+y^2+z^2=1\ , \tag{1}$$

P' is the point $(w,x,y,0)$, P'' is the point $(w,x,0,0)$, and θ,φ,ψ are defined as

$$\theta=P\hat{O}Z,\ \varphi=P'\hat{O}Y \text{ and } \psi=P''\hat{O}X\ . \tag{2}$$

Then the hyper-stereographic projection of P with respect to the south pole $(0,0,0,-1)$ is a point on the hyper-plane $z=0$ with spherical polar coordinates $(\tan\theta/2,\varphi,\psi)$.

It is evident that the hyper-stereographic projection of all points with $\theta<\pi/2$ lies inside or upon a unit sphere, and this constitutes the primitive of the projection. Points with $\theta>\pi/2$ lie outside the primitive and the projection extends indefinitely in all directions. However, by the usual expedient of projecting such points with respect to the north pole $(0,0,0,1)$ the whole projection may for many purposes be confined within the unit sphere.

Properties of the hyper-stereographic projection

It follows from (1) and (2) that

$$\cos\theta=z,\ \cos\varphi=y/(w^2+x^2+y^2)^{1/2} \text{ and } \cos\psi=x/(w^2+x^2)^{1/2}. \tag{3}$$

Thus a point (w,x,y,z) on the hyper-sphere (1) projects to the point

$$(1-z)^{1/2}/(1+z)^{1/2},\ \cos^{-1}[y/(w^2+x^2+y^2)^{1/2}],\ (\omega/|\omega|)\cos^{-1}[x/(w^2+x^2)^{1/2}] \tag{4}$$

in spherical polar coordinates. Equally, subject to condition (1), we can regard w,x,y,z as the direction cosines of a line OP, where P is the point (w,x,y,z), and we can take the coordinates (4) as defining the representation of such a direction in the hyper-stereographic projection.

(*a*) *The representation of the angle between two directions*

Provided that the z axis, which defines the pole of projection, is left unchanged there is no loss of generality if the other three axes are chosen in the most convenient way. In order to consider the projections of two points P_1 and P_2 whose coordinates satisfy (1), we therefore choose the y axis perpendicular to Oz, OP_1 and OP_2. The process of projection is then identical to the ordinary stereographic projection of the sphere

$$w^2+x^2+z^2=1$$

on to the plane $z=0$. Thus the points P_1 and P_2 will project to points P_1' and P_2' on this plane such that the arc P_1P_2 is represented by the arc $P_1'P_2'$ in the terms of the usual metric of the Wulff net. It follows that the arc P_1P_2 can always be measured by superimposing a Wulff net on the unique central section of the hyper-stereogram which contains P_1' and P_2'.

(*b*) *The representation of a hyper-plane*

The hyper-plane

$$kw+lx+my+nz=p \qquad (5)$$

intersects the hyper-sphere (1) in a sphere, T. Because k,l,m,n are the direction cosines of the normal to this hyper-plane, and w,x,y,z are the direction cosines of the radius vector to any point on its intersection with the hyper-sphere, this spherical intersection can be regarded as the locus of points at a constant angular distance $\cos^{-1} p$ from the pole Q of (5) on the hyper-sphere (*cf.* the angular radius of the circular intersection of a plane with a sphere).

Consider any central section of the hyper-stereogram through Q', the projection of Q. Those points of T which project on to this plane will project to points whose arcual distances from Q' are constant (namely $\cos^{-1} p$) in terms of the metric of the Wulff net. The locus of such points is of course well known to be a circle. This is true for all central sections through Q', so that the complete locus is obtained by rotating the circle so defined about its diameter through the origin. Thus any sphere on the hyper-sphere projects to a sphere. The centre of this spherical projection lies on the radius of the hyper-stereogram which passes through the projection of the centre of the sphere* that is to be projected. The radius of the spherical projection and the position of its centre for a given value of $\cos^{-1} p$ are evidently identical with the corresponding values for the projection of a small circle in an ordinary stereographic projection.

In the special case of a central section of the hyper-sphere $p=0$. Just as the projection of a great circle on a stereogram always intersects the primitive at the ends of the diameter perpendicular to the radius vector of its centre, it follows that the projection of a 'great sphere' intersects the primitive of the hyper-stereogram in the great circle that is perpendicular to the corresponding radius vector (Fig. 1).

The following further special cases follow directly. The projection of the great sphere given by the intersection of (1) with

$$kw+lx+my=0 \qquad (6)$$

is a central plane of the hyper-stereogram whose equation is identical with (6), since it is the locus of the projections of points 90° from the projection of the point $(k,l,m,0)$. Also it is evident from (4) that the projection of the great sphere given by the intersection of (1) with $z=0$ is the sphere $r=1$, *i.e.* the primitive.

In the ordinary stereogram it is convenient to describe the projection of a great circle as a representation in the stereogram of the central plane that cuts the sphere in that great circle. This enables us to regard such a projection of a great circle as the representation of a plane of mirror symmetry, for example. In the same way it is convenient to regard the projection of a

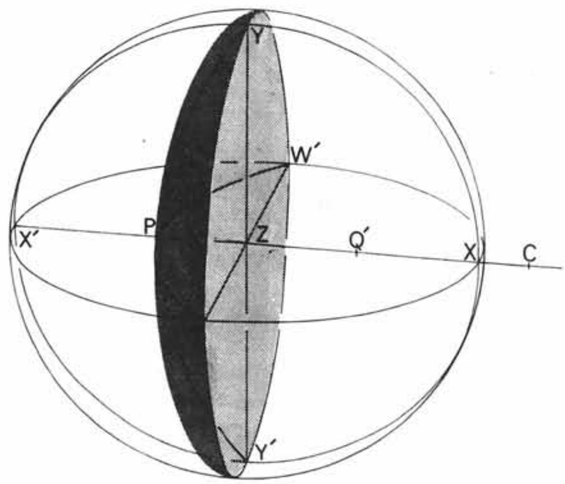

Fig. 1. Hyper-stereographic projection of a great sphere. The centre of the great sphere (*i.e.* the pole of its hyper-plane) projects to Q'. The projection of the great sphere is the spherical surface centred at C. Only that part of it within the primitive is shown; it cuts the WXZ and WYZ planes in the circular arcs WPW' and YPY'. These are identical with the lines representing great circles on Wulff nets on these two planes.

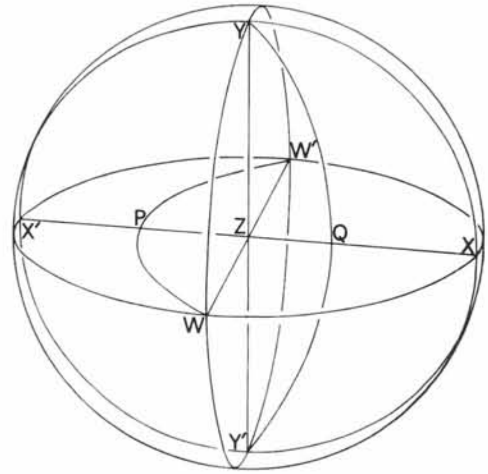

Fig. 2. Hyper-stereographic projection of two absolutely perpendicular planes. The two planes are represented by the circles WPW' and YQY' (only those portions within the primitive are drawn).

* The whole discussion relates to projections of points on the boundary of the hyper-sphere and this centre is therefore to be understood as lying on the boundary of the hyper-sphere in the same sense as the centre of a circle on a sphere is understood to lie on the spherical surface and not at a point in its own plane inside the sphere.

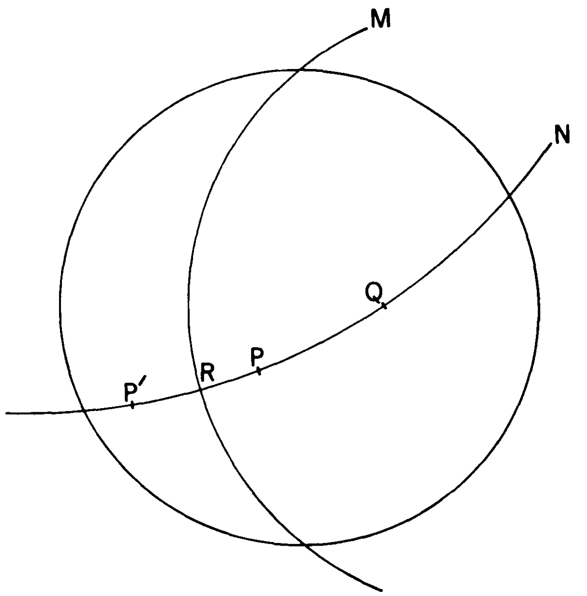

Fig. 3. Ordinary stereographic projection. A plane of mirror symmetry lies on the great circle whose projection is M and whose pole is Q. N is the great circle through P and Q, and it intersects M in R. Then, if $PR = RP'$ in terms of the metric of the Wulff net, P' is the reflexion of P in M.

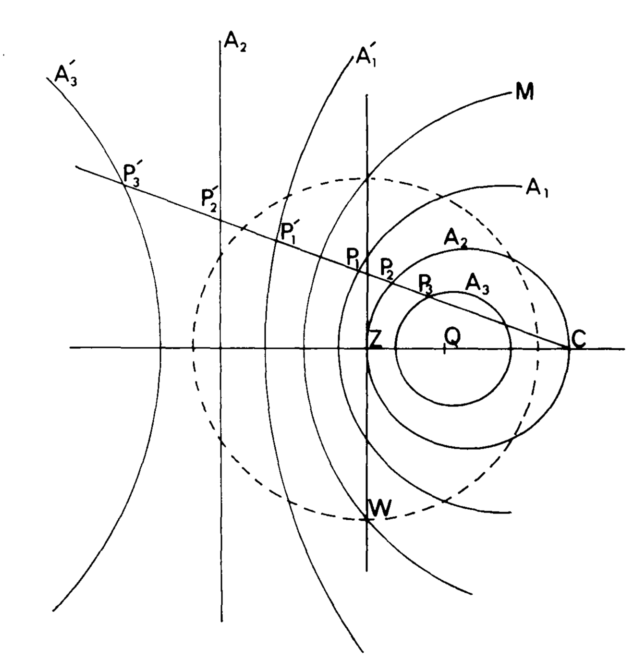

Fig. 4. Ordinary stereographic projection. The primitive is shown with a broken line. A plane of symmetry lies on the great circle whose projection is M and whose pole is Q. C is the centre of the circle M. The circles A_i and A_i' are projections of small circles about Q of radii $90^\circ - \theta_i$ and $90 + \theta_i$ respectively. The reflexion in M of a point P_i on A_i lies at the intersection P_i' of CP_i with A_i'. When a small circle passes through C (A_2) its reflexion A_2' in M is a straight line.

great sphere in the hyper-stereogram as the representation of a central hyper-plane.

(*c*) *The representation of a plane*

Following this convention it is evident that, since a plane is the intersection of two hyper-planes, the representation of a plane in the hyper-stereogram will be the line of intersection of the representations of two such hyper-planes. Since these are always spherical (or planar) their line of intersection will always be a circle (or straight line) of radius equal to or less than the radius of the lesser of the two spheres. Important special cases are:

(i) planes which contain the z axis are represented by diameters of the hyper-stereogram;

(ii) central planes perpendicular to the z axis are represented by great circles of the primitive.

(iii) the central plane perpendicular to both the z axis and the y axis is represented by the great circle of the primitive in the plane WXZ, *etc*;

(iv) any central plane is represented by a circle which passes through opposite ends of a diameter of the primitive;

(v) the representations of two central planes that are absolutely perpendicular* lie in perpendicular planes, and the intersection of either line with the plane of the other is at a point that is 90° from all points on the other in terms of the metric of the Wulff net. (Fig. 2).

The representation of a reflexion

The conventional construction to find the reflexion of a pole P in a general great circle M in the ordinary stereogram is shown in Fig. 3. If Q is the pole of M, then the great circle N through P and Q is drawn to intersect M in R, and a projected angle RP' ($=PR$) is cut off along N. P' is then the reflexion of P.

The same construction may be applied in the hyper-stereogram. If the mirror hyper-plane is

$$kw + lx + my + nz = 0$$

there is no loss of generality if we transform the axes so that the hyper-plane contains the wy plane. Its equation then becomes

$$lx + nz = 0 \; . \tag{7}$$

We may then further rotate the axes about the xz plane without modifying (7) so that the y coordinate of the point to be reflected becomes zero. The problem is thus reduced to a three-dimensional one and may be solved by applying the method of Fig. 3 on the central plane of the hyper-stereogram perpendicular to YZ.

* In four-dimensional geometry two planes are said to be absolutely perpendicular if every line in one is orthogonal to every line in the other. Two absolutely perpendicular planes intersect in only a single point. To every central plane there is one, and only one, absolutely perpendicular central plane.

However, the effect of a reflexion in the hyper-stereogram is further clarified by the following consideration:

In Fig. 4 let M be the ordinary stereographic projection of the plane

$$x \sin \beta + z \cos \beta = 0 \,. \tag{8}$$

Reflexion in this plane corresponds to the transformation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\beta & -\sin 2\beta \\ 0 & -\sin 2\beta & -\cos 2\beta \end{pmatrix} \begin{pmatrix} w \\ x \\ z \end{pmatrix} = \begin{pmatrix} w \\ x \cos 2\beta - z \sin 2\beta \\ -x \sin 2\beta - z \cos 2\beta \end{pmatrix} . \tag{9}$$

We may then find the plane through a point $P(w,x,z)$, its reflexion given by (9), and the south pole $(0,0,-1)$. This plane may be shown to intersect the x axis at $x = \tan \beta$, and this is the centre of circle M. Thus reflexion involves a movement along the radius of M, from a given small circle about Q within M to the corresponding small circle about Q outside M, or *vice versa*, as shown in Fig. 4.

Since it has been shown above that the central plane of the hyper-stereogram perpendicular to YZ is identical with Fig. 4, and since the projection of the hyper-plane (7) is axially symmetric about the line XZ in the hyper-stereogram (Fig. 1), it follows that reflexion in (7) is equivalent to a movement along a radius of the sphere S to which (7) projects, from a small sphere about the pole of (7) inside S to the corresponding small sphere outside S, or *vice versa*. These nested spheres are obtained by rotating Fig. 4 about XZ.

This way of regarding a reflexion provides a simple transition to the special cases. When the mirror hyper-plane is perpendicular to the z axis its projection is the primitive, and a projected reflexion involves a radial movement from a point inside the primitive to an equal distance outside (or *vice versa*) in terms of the metric of the stereographic ruler. If a point inside the primitive is reflected in this way to one outside the primitive, and this point is replotted by projection from the north pole, it then coincides with the original point in the usual way. When the mirror hyper-plane contains the z axis then its projection is a central plane of the hyper-stereogram, and a projected reflexion is a simple reflexion in this plane.

The representation of a four-dimensional rotation

One of the most useful features of the stereogram is the ease with which the effect of a rotation can be represented on it. It is therefore desirable to investigate the corresponding use of the hyper-stereogram, especially in view of the conceptual difficulty that arises in four dimensions from the fact that rotations take place about planes and not about axes.

The simplest case to consider is rotation about the yz plane through an angle α. The effect of this is to move every point $P_1(w_1,x_1,y_1,z_1)$ to $P_2(w_2,x_2,y_2,z_2)$ defined by

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \cos\alpha + x_1 \sin\alpha \\ -w_1 \sin\alpha + x_1 \cos\alpha \\ y_1 \\ z_1 \end{pmatrix} .$$

Since $y_2 = y_1$ and $z_2 = z_1$ we have

$$w_2^2 + x_2^2 + y_2^2 = w_1^2 + x_1^2 + y_1^2$$

and

$$w_2^2 + x_2^2 = w_1^2 + x_1^2 \,.$$

It follows from (4) that if the projections of P_1 and P_2 are $P_1'(r_1,\varphi_1,\psi_1)$ and $P_2'(r_2,\varphi_2,\psi_2)$ then

$$r_2 = r_1,\ \varphi_2 = \varphi_1, \text{ and } \psi_2 = \psi_1 + \alpha \,.$$

Thus the effect is a simple rotation of all points of the hyper-stereogram about the diameter which represents the rotation plane.

If the rotation plane is perpendicular to the z axis, it is convenient to take the y axis also perpendicular to it. The rotation plane is then represented by the great circle of the primitive in the WX plane. The transformation is now

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ x_1 \\ y_1 \cos\alpha + z_1 \sin\alpha \\ -y_1 \sin\alpha + z_1 \cos\alpha \end{pmatrix} .$$

Hence from (4) $\psi_2 = \psi_1$, so that P_2' always lies in the plane through P_1' and the y axis. Let this plane cut the XY great circle of the primitive in R' (Fig. 5). Then this is the projection of the point $R(w_1,x_1,0,0)$, and P_1,P_2 are equidistant from R. Accordingly P_1',P_2' are equidistant from R' in terms of the Wulff metric on the plane $P_1'P_2'R'$. In other words P_1' is transformed to P_2' by a movement of α along a small circle around R' defined in the usual way on the Wulff net on this plane. The effect of the rotation on the hyper-stereogram is thus a sort of toroidal rotation about a great circle, although only points infinitesimally close to the great circle move in paths centred on it, and the further the point P_1' is removed from the great circle the greater is the outward displacement of its centre of rotation.

If the rotation plane neither contains nor is perpendicular to the z axis then the rotation leaves none of the coordinates r, φ, ψ unchanged. However, the problem is conveniently solved by a similar consideration to that used in connexion with reflexions.

Let the axes be chosen so that the rotation plane S contains the w axis and is perpendicular to the y axis and to the vector in the xy plane which makes an angle β with the z axis. Its projection is then a circle on the plane WXZ centred at C on ZX (produced if necessary)

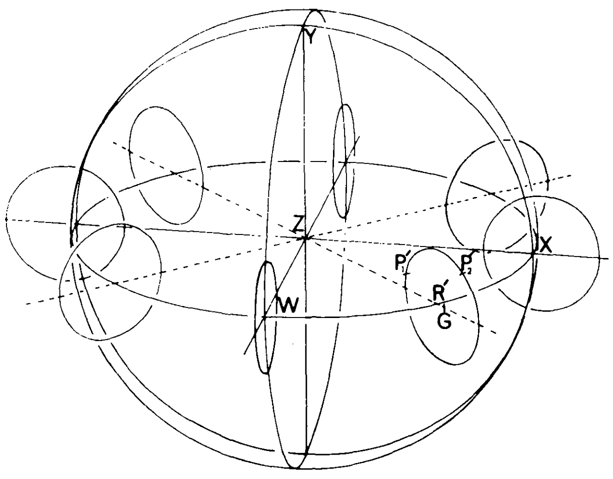

Fig. 5. Hyper-stereographic projection showing the effect of a rotation about the wx plane. The point P_1' moves in the plane YZP_1' along a small circle of the Wulff net about R', the centre of this circle being G. Corresponding loci are drawn in various other planes for points at the same distance as P_1' from the wx plane, in order to indicate the toroidal character of the rotation.

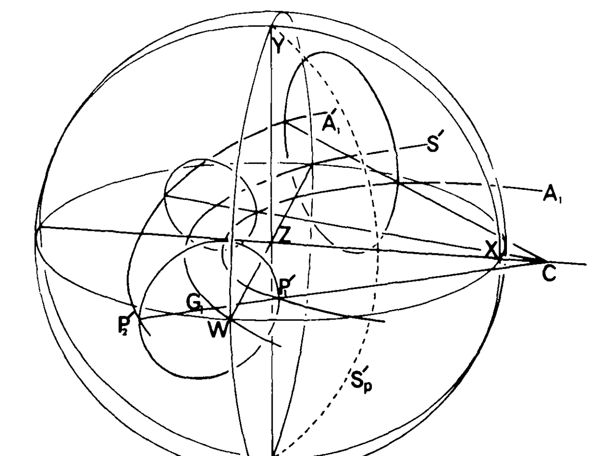

Fig. 6. Hyper-stereographic projection showing the effect of a rotation about the plane represented by the circle S', centre C. S_p' represents the plane absolutely perpendicular to S. The locus of P_1' for a rotation about S is the circle centred at G in the plane through CP_1' parallel to YZ. Three such loci are shown for points at the same angular distance θ from S but having different coordinates. They all lie between the small circles A_1, A_1' that are on the WXZ plane and have radii $90° \pm \theta$ about the intersection of S_p' with XZ.

such that $ZC = \tan\beta$. The matrix (R_β) for rotation through an angle α about this plane may be constructed from the matrix (R) for rotation about an axial plane and the matrix (M) for rotation of the chosen axes to those which make the chosen plane axial

$$i.e. \quad \mathsf{R}_\beta = \mathsf{M}^{-1}\mathsf{R} \,.\, \mathsf{M}\,.$$

The position vector $\mathbf{v}_1$ of a point P_1' (w_1, x_1, y_1, z_1) may then be transformed to give

$$\mathbf{v}_2 = \mathsf{R}_\beta \,.\, \mathbf{v}_1 \quad \text{and} \quad \mathbf{v}_3 = \mathsf{R}_\beta^{-1} \,.\, \mathbf{v}_1\,.$$

The equation of the hyper-plane through the corresponding points P_1, P_2, P_3 and $(0,0,0,-1)$ can then be calculated and shown to be parallel to the y axis and to intersect the x axis at $x = \tan\beta$.

This hyper-plane will necessarily intersect the hyper-plane $z=0$ in P_1', P_2', P_3', the projections of P_1, P_2, P_3. Thus the plane defined by P_1', P_2', P_3' (containing the locus of P_1' under rotation) is parallel to YZ and contains the point C.

Since the locus of P under rotation is a circle so is the locus of its projection P', but as it lies on a non-central plane of the hyper-stereogram it cannot be simply described in terms of a Wulff net. The position of the circular locus and the metric along its circumference may be obtained as follows:

The point P and the rotation plane S together define a hyper-plane H that is represented by a sphere H' on which S' lies. When P rotates about S so does H, and the normal to H at the origin describes the central plane S_p that is absolutely perpendicular to S. Thus the pole Q of H' moves along S_p' which is a circle in the XYZ plane of the hyper-stereogram (Fig. 6). Since this is a circle on a central section of the hyper-stereogram, its metric is defined by the Wulff net. If P_1' lies in the WXZ plane then H_1' is represented by the WXZ plane and Q_1' is coincident with Y. Let P_2 be related to P_1 by 180° rotation about S. Then Q_2' will be coincident with Y, H_2' will again coincide with the OWX plane, and P_2' will therefore also lie in the WXZ plane. Hence P_1' and P_2' are equally but oppositely displaced from S' along the line CP_1' in accordance with the metric of Fig. 4. Thus if P is distant θ from S, the centre G of the circular locus of P' lies midway between A_1 and A_1' (Fig. 6) which are small circles of radii $90° \pm \theta$ about the point of intersection of S_p' and XZ.

Three examples of such circular loci are shown in Fig. 6. The metric around the loci is defined as follows for a rotation α. For a given initial point P_1' construct the sphere H_1'. Find the position of the pole Q_1' of H_1' on S_p'. Construct the point Q_2'' at angle α from Q_1' S_p' in accordance with the metric of the Wulff net. Construct the sphere H_2' having Q_2' as its pole. The intersection of H_2' with the circular locus through P_1' defines P_2'.

Thus the effect of rotation about a general plane is again represented in the hyper-stereogram by a sort of toroidal rotation about the projection of that plane, although the sizes of the circular loci are no longer constant for a given angular displacement from the rotation plane. This result reduces to the kind of toroidal rotation discussed previously when the projection of the rotation plane lies on the primitive, and to an ordinary rotation when it is a diameter of the primitive.

Conclusion

The properties of the hyper-stereogram that have been described provide the necessary groundwork for its use in the description of four-dimensional crystallographic symmetry, and this will be discussed in the following paper (Whittaker, 1973).

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WHITTAKER, E. J. W. (1973). *Acta Cryst.* **A29**, 678–684.

Acta Cryst. (1973). A**29**, 678

A Representation of Hyper-Cubic Symmetry and its Projections

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(*Received* 5 *March* 1973; *accepted* 1 *April* 1973)

The symmetry elements of the four-dimensional hyper-cube are represented in a hyper-stereogram. This is used to classify the special and general directions in holosymmetric hyper-cubic symmetry. Projections of the hyper-cube in these various directions are constructed, with the incorporation of colour perspective, and the two-colour Shubnikov symmetry of these projections is tabulated and related to the four-dimensional symmetry elements on which the projection directions lie. The general representation of other four-dimensional symmetry elements in the hyper-stereogram is discussed, and the convenience of the hyper-stereogram for facilitating the evaluation of the matrices of symmetry operations in non-standard orientations is demonstrated.

There has recently been much development in the theoretical understanding of four-dimensional crystallography and in the derivation of the four-dimensional crystal classes (*e.g.* Belov & Kuntsevich, 1971; Neubüser, Wondratschek & Bülow, 1971*a*, *b*, *c* and references therein). This work has been done in terms of the matrix representations of symmetry operations, but it is of some interest to be able to visualize the geometrical relationships of the corresponding four-dimensional symmetry elements just as we do in three dimensions, especially for the higher-symmetry crystal classes. The possibility of doing this conveniently is provided by the hyper-stereogram (Whittaker, 1973).

1. Nomenclature

The nomenclature of four-dimensional symmetry operations has hitherto been in terms of arbitrary letters (Hurley, 1951), the parameters of the characteristic equation of their matrices (Hurley, 1951), a sequence of up to four symbols representing the multiplicities of their irreducible components (Hermann, 1949), or a pair of (Cyrillic) letters indicating the general nature of the operation qualified by numerical subscripts (Kuntsevich & Belov, 1968). Since the numerical subscripts of Kuntsevich & Belov are all different they suffice as symbols in their own right. This not only simplifies the nomenclature, but also clarifies the relationship of the four-dimensional symmetry operations and elements to those in three dimensions.

In order to avoid ambiguity in this relationship it is however necessary to make two minor changes. The subscripts $\bar{n}$ of Kuntsevich & Belov are combinations of an n-fold rotation with a mirror reflexion, and are related to three-dimensional n-fold rotation–reflexion axes, not n-fold rotation–inversion axes. They have therefore been changed in this paper to $\tilde{n}$, and this nomenclature is also used for three-dimensional rotation–reflexion axes to distinguish them from $\bar{n}$-rotation–inversion axes. The other change arises because Kuntsevich & Belov retained Hermann's nomenclature of 5 and 10 for the pentatope, and a related, operation. To make clear the distinction from the non-crystallographic fivefold and tenfold rotation planes these have been replaced by V and X.

2. The hyper-stereogram of the hyper-cubic holosymmetry

The hyper-stereogram is shown, as a stereo pair, in Fig. 2. The nomenclature of the axes of Fig. 2 is shown in Fig. 1. Planes of rotation symmetry project in the